Lec 4

I Matrix Simplified

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

An $m \times n$ matrix is an array of mn entries, called elements arranged in m rows and n columns.

a_{ij} called the ij^{th} entry is the element in its i^{th} row and j^{th} column.

Ex 4·1

1×1 matrix $\begin{pmatrix} 5 \end{pmatrix}$

3×4 matrix $\begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}$

2×2 matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

1×3 matrix $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

3×1 matrix $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Square matrix:

when $m = n$

Matrix addition
Matrix scalar multiplication $\Big\}$ usual.

A^T will denote transpose of a matrix

If $B = A^T$

then we define $b_{ij} = a_{ji}$.

B is a $n \times m$ matrix.

B is obtained from A by interchanging rows and columns.

Ex:

$$A = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$

Matrix multiplication:

If A is a $m \times n$ matrix

B is a $n \times p$ matrix

we define

$$C = AB$$

where

C is a $m \times p$ matrix

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Ex

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 9 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 9 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 3 \end{pmatrix}$$

$$= \left(\begin{array}{c|c|c} 1\cdot 0 + 2\cdot 1 + 5\cdot 2 & 1\cdot 1 + 2\cdot 0 + 5\cdot 1 & 1\cdot 0 + 2\cdot 0 + 5\cdot 3 \\ \hline 2\cdot 0 + 3\cdot 1 + 9\cdot 2 & 2\cdot 1 + 3\cdot 0 + 9\cdot 1 & 2\cdot 0 + 3\cdot 0 + 9\cdot 3 \end{array}\right)$$

$$= \begin{pmatrix} 12 & 6 & 15 \\ 21 & 11 & 27 \end{pmatrix}$$

In the previous example, although AB is defined BA is not defined.

Ex
$$A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 9 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 9 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 6 \\ 21 & 11 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 9 \\ 1 & 2 & 5 \\ 4 & 7 & 19 \end{pmatrix}$$

Here AB and BA are both defined but they are of different sizes.

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Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 5 \\ 3 & 9 \end{pmatrix}$

$$AB = \begin{pmatrix} 8 & 23 \\ 13 & 37 \end{pmatrix}$$

$B = \begin{pmatrix} 2 & 5 \\ 3 & 9 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

$$BA = \begin{pmatrix} 12 & 19 \\ 21 & 33 \end{pmatrix}$$

Here AB and BA are both defined, are of the same sizes but are not equal.

Caution:

When we are multiplying two matrices A & B, it is not necessarily true that $AB = BA$ even when both AB and BA are defined and are of the same order.

$$AB \neq BA \text{ in general.}$$

Good News:

Whenever the operations are defined we have

① $(AB)C = A(BC) = ABC$

② $A(B+C) = AB + AC$.

③ $(A+B)C = AC + BC$

④ $(\lambda A)B = A(\lambda B) = \lambda AB$.

⑤ $(AB)^T = B^T A^T$.

6 $(A+B)^T = A^T + B^T$.

Note: $AB = 0 \not\Rightarrow$ Either A or B is 0.

Ex: $A = \begin{pmatrix} 1 & 4 \\ 3 & 12 \end{pmatrix}$ $B = \begin{pmatrix} 4 & -8 \\ -1 & 2 \end{pmatrix}$

$AB = 0$.

Zero Matrix: A matrix with all zero entries

Identity Matrix: A square matrix with 1 in the diagonal and zero elsewhere

Ex: 2x2 zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

2x2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

If A is any 2x2 matrix then

$A \cdot I = I \cdot A = A$.

Let A be a square $n \times n$ matrix.

$$\text{trace } A = \sum_{i=1}^{n} a_{ii} \quad \text{(sum of the diagonal elements)}$$

A is symmetric if $a_{ij} = a_{ji} \ \forall i, j$.

A is skew symmetric if $a_{ij} = -a_{ji} \ \forall i, j$.

A is orthogonal if $A A^T = A^T A = I$.

$$A = \begin{pmatrix} 1 & 5 & 9 \\ 5 & 2 & 8 \\ 9 & 8 & 5 \end{pmatrix}$$ is a symmetric matrix.

$$A = \begin{pmatrix} 0 & 5 & 9 \\ -5 & 0 & 8 \\ -9 & -8 & 0 \end{pmatrix}$$ is a skew symmetrix matrix.

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$ is an orthogonal 2×2 matrix.

II The trace and the determinant — characteristic polynomial:

Ex: $$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\text{trace } A = a_{11} + a_{22}$$

$$\det A = a_{11}\, a_{22} - a_{12}\, a_{21}$$

$$\left.\begin{aligned} m_{11} &= a_{22} \\ m_{12} &= a_{21} \\ m_{21} &= a_{12} \\ m_{22} &= a_{11} \end{aligned}\right\} \boxed{\text{minors}}$$

$$\begin{pmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ \cancel{a_{21}} & a_{22} \end{pmatrix}$$

$$c_{ij} = (-1)^{i+j} m_{ij} \leftarrow \text{co-factors.}$$

$$c_{11} = m_{11} = a_{22}$$

$$c_{12} = -m_{12} = -a_{21}.$$

$$c_{21} = -m_{21} = -a_{12}.$$

$$c_{22} = m_{22} = a_{11}$$

$$\operatorname{adj} A = \begin{pmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$A \cdot \operatorname{adj} A = \operatorname{adj} A \cdot A = \begin{pmatrix} \det A & 0 \\ 0 & \det A \end{pmatrix}$$

It follows that if we define

$$B = \frac{1}{\det A} \operatorname{adj} A \quad (B \text{ is defined if } \det A \neq 0)$$

we have

$$AB = BA = I$$

The matrix B is called A^{-1},
"A inverse"

and we have

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

$$= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Many often $\det A$ is also written as $|A|$.

Ex

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 5 & 3 & 0 \\ 2 & 4 & -1 \end{pmatrix}$$

$$m_{11} = \begin{vmatrix} 3 & 0 \\ 4 & -1 \end{vmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 5 & 3 & 0 \\ 2 & 4 & -1 \end{pmatrix}$$

$$m_{12} = \begin{vmatrix} 5 & 0 \\ 2 & -1 \end{vmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 5 & 3 & 0 \\ 2 & 4 & -1 \end{pmatrix}$$

The cofactor matrix $C = (c_{ij})$ is given by

$$C = \begin{pmatrix} -3 & 5 & 14 \\ 4 & -3 & -4 \\ -3 & 5 & 3 \end{pmatrix}$$

$$\operatorname{adj} A = \begin{pmatrix} & & \\ & C & \\ & & \end{pmatrix}^T = \begin{pmatrix} -3 & 4 & -3 \\ 5 & -3 & 5 \\ 14 & -4 & 3 \end{pmatrix}$$

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$A \cdot \operatorname{adj} A =$

$$\begin{pmatrix} 1 & 0 & 1 \\ 5 & 3 & 0 \\ 2 & 4 & -1 \end{pmatrix} \begin{pmatrix} -3 & 4 & -3 \\ 5 & -3 & 5 \\ 14 & -4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$

$\det A = 11 \neq 0$

$$A^{-1} = \frac{1}{\det A} \operatorname{Adj} A$$

$$= \begin{pmatrix} -3/11 & 4/11 & -3/11 \\ 5/11 & -3/11 & 5/11 \\ 14/11 & -4/11 & 3/11 \end{pmatrix}$$

Characteristic polynomial :-

Let A be any square matrix

We define a polynomial

$$p(\lambda) = \det(\lambda I - A)$$

and call $p(\lambda)$ the characteristic polynomial of A.

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$p(\lambda) = \det \begin{pmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{pmatrix}$$

$$= (\lambda - 1)(\lambda - 4) - 3 . 2$$

$$= \lambda^2 - 5\lambda + 4 - 6$$

$$= \lambda^2 - 5\lambda - 2$$

Note that

trace $A = 5$

det $A = 4 - 6 = -2$.

It appears that

$$\boxed{p(\lambda) = \lambda^2 - (\text{trace } A)\lambda + (\det A)}$$

Ex: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\boxed{p(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})}$$

Ex:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \end{pmatrix}$$

$$\underbrace{p(\lambda) = \lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0}_{\text{A little algebra will show this.}}$$

$$p(\lambda) = \lambda^4 - (\text{trace } A)\lambda^3 + * \lambda^2 + * \lambda + (\det A)$$

III Equation Solving, Cramer's Rule :

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Solution of the system of equations

$$a_{11} x + a_{12} y + a_{13} z = b_1$$
$$a_{21} x + a_{22} y + a_{23} z = b_2$$
$$a_{31} x + a_{32} y + a_{33} z = b_3$$

$$A \underline{x} = b$$
$$\Leftrightarrow \underline{x} = A^{-1} b = \frac{1}{\det A} \operatorname{adj} A \cdot b$$

$$x = \frac{\det A_1}{\det A}$$

$$y = \frac{\det A_2}{\det A}$$

$$z = \frac{\det A_3}{\det A}$$

Cramer's Rule

A_i is the matrix A with its ith column replaced by b.

<u>Remark</u>:

$$A_1 = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}$$

$$\det \cdot A_1 = b_1 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - b_2 \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$+ b_3 \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Likewise one can calculate

$\det A_2$ and $\det A_3$.

It is easy to see by explicit calculation that

$$\operatorname{adj} A \cdot b = \begin{pmatrix} \det A_1 \\ \det A_2 \\ \det A_3 \end{pmatrix}$$

IV Cayley Hamilton Theorem :

"Every square matrix satisfies its own characteristic polynomial equation"

Consider Example on page 4·16.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$p(\lambda) = \lambda^2 - 5\lambda - 2$$

Cayley Hamilton says that

$$p(A) = 0$$

ie $A^2 - 5A - 2I = 0$

or $\boxed{A^2 = 5A + 2I}$

We would now like to verify this

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

$$5A = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix} \qquad 5A + 2I = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

$$2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

— x —

In general if A is a $n \times n$ matrix.
C. H. says that A^n can be
written as a l. c. of $I, A, A^2, \dots, A^{n-1}$.

Ex:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -9 & -8 & -2 & -1 & 13 \end{pmatrix}$$

$$p(\lambda) = \lambda^5 - 13\lambda^4 + \lambda^3 + 2\lambda^2 + 8\lambda + 9$$

C.H. would say that

$$\boxed{A^5 = 13A^4 - A^3 - 2A^2 - 8A - 9I}.$$

I would not dare verifying this by hand.